international journal for
SCIENTIFIC RESEARCH INNOVATIONS

Volume 1, Issue 1
May 2023, pp. 1-6
Date of Publication: 29/05/2023

Generalized Functions and Distribution

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***Abstract* - Generalized functions, also known as distributions, provide a powerful mathematical framework for dealing with objects that do not fit into the traditional notion of functions. Generalized functions is defined as linear functionals acting on a space of test functions. The linearity property of generalized functions is emphasized, showcasing their ability to handle superposition of values. Continuity is discussed as another important property of generalized functions. The concept of convolution of distributions enables the combination of the effects of two distributions through an integral operation. Applications of generalized functions are in various fields of mathematics and physics.**

***Keywords* – Generalized Functions, Neutrix Product, Dirac Delta Function.**

1. Introduction

Generalized functions, also known as distributions, are mathematical objects that extend the concept of functions to a broader class of objects that may not be traditional functions in the classical sense. They provide a rigorous framework for dealing with objects such as point masses, impulses, and singularities that do not fit into the traditional notion of functions.

It is a linear function over a space containing test functions, usually smooth functions, that perform well with compact support. Its action on a test function is given by an integral or limit involving the product of the generalized function and the test function. This allows the generalized function to be applied to the test function and produce a result, which can be a number or another function. The key idea behind generalized functions is that they can act as "distributions" or "weights" that assign values to test functions, and they do so in a way that extends the concept of pointwise evaluation of functions.

Laurent Schwartz laid the foundation of distribution theory when he introduced the distribution theory as a rigorous framework to handle Dirac delta function (DDF). Schwartz's work provided a formal definition and algebraic operations for distributions, establishing a mathematical structure that extended beyond the realm of classical functions.

The motivation behind the study of generalized functions and distribution theory is rooted in the need for a comprehensive mathematical framework that can accurately represent and manipulate a wider class of mathematical objects. These objects may include point sources, impulses, discontinuities, and singularities, which arise in diverse disciplines such as physics, engineering, and signal processing. Generalized functions and distributions provide a systematic approach to handle such mathematical entities, allowing researchers to solve complex problems more effectively and rigorously.

Moreover, distribution theory plays a fundamental role in many branches of mathematics and mathematical physics. It serves as a powerful tool in the analysis of partial differential equations, where solutions may exhibit singular behavior or lack conventional smoothness. Distribution theory also finds extensive applications in Fourier analysis, quantum mechanics, general relativity, and other fields where the behavior of non-smooth objects needs to be understood and modeled accurately.

Colombeau and Meril [1] in their research on generalized functions and multiplication of distribution on ζ^{x} manifold introduced an improvement in the definition that permitted them to make invariant within the action of ζ^{x} diffeomorphisms technique of generalized functions for defining the arbitrary products of distribution.

Kunzinger et al [2] developed the generalized differential geometry theory and considered the space $G[X, Y]$ of generalized functions of Colombeau which is defined on a manifold X while considering the values in Y manifold. They also introduced an embedding of space of continuous mappings $C(X, Y)$ into $G[X, Y]$ and studied the sheaf properties of $G[X, Y]$. They concluded that same outcomes were achieved for spaces of generalized vector bundle homomorphisms.

Shantanu Dave [3] in his research used spectral theory for producing embeddings of distributions into algebras of generalized functions on a closed Riemannian manifold. Through this he found that the embeddings are invariant under isometries and preserve the singularity structure of the distributions.

Shamilov et al [4] in their research applied the concept of generalized probability density function based on generalized Heaviside and Dirac functions for transforming random variables to obtain the distribution of several random variables. discrete and continuously transformed. They also found a theorem to obtain the general distribution of a continuously varying random system and some of its applications.

Chalishajar et al [5] in their study discussed the solid mechanics model based on Schwartz distribution theory for beam bending differential equations. This problem is solved using general functions, including the famous DDF. The governing differential equation is the equation of an Euler-Bernoulli beam with discontinuous jumps during translation and rotation.

The primary objective of studying generalized functions and distribution theory is to develop a comprehensive understanding of this mathematical framework and its practical applications. By exploring the foundational concepts, properties, and operations of distributions, researchers can gain insights into the mathematical structure underlying generalized functions. Furthermore, the research aims to investigate the applications of distribution theory in various fields, highlighting its significance in solving complex mathematical problems and addressing real-world phenomena.

2. Properties of Generalized Function

Generalized functions have several important properties that make them valuable tools in mathematical analysis and physics. For example:

- **Linearity:** Generalized functions are linear, meaning that they satisfy the property of superposition. If a generalized function assigns a value to one test function and another generalized function assigns a value to a different test function, their linear combination will assign a corresponding linear combination of values to the linear combination of test functions.

- **Continuity:** Generalized functions can be continuous, meaning that they possess a certain degree of regularity and stability with respect to the test functions. This property ensures that small changes in the test functions result in small changes in the assigned values by the generalized function.

- **Convolution:** Generalized functions can be convolved with test functions, leading to the concept of convolution of distributions. The convolution of two distributions is a new distribution obtained by combining the effects of both distributions through an integral operation.

- **Differentiation:** Generalized functions can be differentiated, allowing for the concept of derivatives of distributions. The derivative of a distribution is defined through a limiting process involving differentiation of the corresponding test functions.

Generalized functions are applicable to areas of physics and mathematics. They are used to solve differential equations involving singularities, describe the behavior of physical quantities in the presence of impulses or point sources, analyze signal processing and wave propagation, and provide a rigorous framework for solving problems involving non-classical functions.

3. Neutrix Product and its properties

The Neutrix product is an extended multiplication operation concept to a general distribution or function. It allows multiplying 2-distributions in a clear-cut way, even if their product is not well-defined due to singularities or other problems.

Following are its properties:

- **Commutativity:** The Neutrix product is commutative, meaning that for any two distributions u and v, we have $u \star v = v \star u$.
- **Distributivity over addition:** The Neutrix product distributes over addition, which means that for distributions u, v, and w, we have $(u+v) \star w = u \star w + v \star w$.
- **Associativity:** The Neutrix product is associative, implying that for distributions u, v, and w, we have $(u \star v) \star w = u \star (v \star w)$.
- **Compatibility with scalar multiplication:** The Neutrix product is compatible with scalar multiplication, meaning that for a distribution u and a scalar α, we have $(\alpha u) \star v = \alpha(u \star v) = u \star (\alpha v)$.
- **Zero product property:** If either u or v is a compactly supported distribution, then their Neutrix product $u \star v$ is identically zero.
- **Associativity with convolution:** The Neutrix product is closely related to convolution. Specifically, the Neutrix product of two distributions u and v is equal to the convolution of their regularizations, denoted as $u \star v = u_r * v_r$, where u_r and v_r are the regularizations of u and v, respectively.

- **Compatibility with derivative:** The Neutrix product is compatible with differentiation. If u is a distribution and v is a smooth function, then the Neutrix product of their derivatives is given by (u′ ⋆ v) = u ⋆ (v′) − u(v(0)), where u′ and v′ are the derivatives of u and v, respectively.

These properties highlight the algebraic behavior and compatibility of the Neutrix product with various operations involving distributions. The Neutrix product is a useful tool in certain mathematical contexts, such as the theory of generalized functions, where the product of distributions is not well-defined by conventional means.

4. Dirac Delta Function and its properties

The Dirac delta function, denoted by δ(x), is a mathematical distribution or generalized function that is widely used in mathematics, physics, and engineering. It is named after the physicist Paul Dirac, who introduced the concept in the 1920s. The DDF is characterized by its defining property, which can be informally stated as:

$$\int f(x)\delta(x)dx = f(0)$$

where f(x) is a suitably well-behaved function. This property implies that the DDF acts as an ”identity” under integration, picking out the value of the function at the point x = 0.

Following are the properties of DDF:

Scaling Property: The DDF exhibits the property of scaling, which states that for any nonzero constant a, we have

$$\delta(ax) = \frac{1}{|a|}\delta(x)$$

This property shows that the DDF scales inversely with the magnitude of the scaling factor.

Translation Property: The DDF also possesses the property of translation, which states that δ (x − a) is non-zero only when x = a. This property allows for shifting the point of singularity of the DDF along the real line.

Sifting Property: The sifting property of the DDF states that

$$\int_{-\infty}^{\infty} f(x)\delta(x-a)dx = f(a)$$

where f(x) is a continuous function. This property indicates that the DDF acts as an idealized “sampling” or “picking out” function, evaluating the function f(x) at the point of singularity x = a.

Integral of the Product Property: If f(x) is a continuous function, then

$$\int_{-\infty}^{\infty} f(x)\delta(x)dx = f(0)$$

This property shows that the DDF can be used to evaluate the value of a continuous function at the singularity point x = 0.

Derivative Property: The derivative of the DDF, denoted as δ ′(x), is defined as the distributional derivative. It satisfies the property

$$\int_{-\infty}^{\infty} f(x)\delta'^{(x)}dx = -f'(0)$$

where f(x) is a continuously differentiable function.

Multiplication Property: Multiplying a function f(x) by the DDF δ (x) gives the value of the function at the singularity point, i.e., f(x) δ (x) = f(0) δ (x).

5. Test Function

A test function, also called a smooth compactly supported function, is a well-behaved function that is used to define and evaluate generalized functions. Test functions are typically infinitely differentiable and have compact support.

6. Distributional Derivative

The distributional derivative of a generalized function is an operation that extends the concept of differentiation to distributions. It is defined as the action of the derivative operator on a test function. The distributional derivative allows for the differentiation of generalized functions, even when they are not classically differentiable.

7. Support

The support of a generalized function is the complement of the largest open set on which the function vanishes identically. It represents the set of points where the generalized function has non-zero values or singularities.

8. Convolution

The mathematical process of mixing the effects of two general functions to produce a new general function is known as convolution. It is described as the integral of the cross product of a function and its scaled and translated counterpart. In distribution theory, convolution is a fundamental operation that is used to describe a number of operations and characteristics of generic functions.

In generic functional theory, distributed convolution is a crucial procedure. This enables the effects of two distributions to be combined to create a new distribution. The notion of test functions and distributive operations on these functions are used to define the convolution operation.

9. Regularization

Regularization is a process of approximating a generalized function by a sequence of smooth functions. It is often used to define and evaluate certain operations on distributions that may be ill-defined or not directly applicable to the original generalized function. Regularization allows for the manipulation and calculation of properties of generalized functions using classical function techniques.

10. Conclusion

In conclusion, generalized functions offer a flexible and powerful mathematical tool for addressing situations where traditional functions fall short. Their linearity, continuity, convolution, and differentiation properties, along with their wide range of applications, make them indispensable in many areas of mathematics and physics. The study invites further exploration and study of generalized functions and their role in solving complex problems.

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